

# Designing a Low-Pass Filter

Consider a filter function which is continuous and Gaussian. The weights are given by

$$w(x, y | x_0, y_0) = A e^{-\left(\frac{x-x_0}{\sigma}\right)^2} e^{-\left(\frac{y-y_0}{\sigma}\right)^2} \quad (1)$$

The constant  $A$  is found based on the requirement that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y | x_0, y_0) dx dy = 1$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A e^{-\left(\frac{x-x_0}{\sigma}\right)^2} e^{-\left(\frac{y-y_0}{\sigma}\right)^2} dx dy = 1$$

$$A \int_{-\infty}^{\infty} e^{-\left(\frac{x-x_0}{\sigma}\right)^2} dx \int_{-\infty}^{\infty} e^{-\left(\frac{y-y_0}{\sigma}\right)^2} dy = 1$$

$$A \sigma \sqrt{\pi} \sigma \sqrt{\pi} = 1$$

$$A = \frac{1}{\sigma^2 \pi}$$

$$w(x, y | x_0, y_0) = \frac{1}{\sigma^2 \pi} e^{-\left(\frac{x-x_0}{\sigma}\right)^2} e^{-\left(\frac{y-y_0}{\sigma}\right)^2} \quad (2)$$

Now compute the response function assuming a wave field given by

$$f(x, y) = e^{i(kx + dy)}$$

where  $k$  and  $d$  are wave numbers. The filtered function at  $(0, 0)$

$$\hat{f}(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma^2 \pi} e^{-\left(\frac{x}{\sigma}\right)^2} e^{-\left(\frac{y}{\sigma}\right)^2} e^{i(kx + dy)} dx dy \quad (3)$$

Expand  $e^{i(kx+ly)} = \cos(kx+ly) + i \sin(kx+ly)$   
 $= \cos kx \cos ly - \sin kx \sin ly + i \sin kx \cos ly + i \cos kx \sin ly$

Now,

$$\hat{f}(0,0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\left(\frac{x}{\sigma}\right)^2} e^{-\left(\frac{y}{\sigma}\right)^2} \cos kx \cos ly \, dx \, dy - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\left(\frac{x}{\sigma}\right)^2} e^{-\left(\frac{y}{\sigma}\right)^2} \sin kx \sin ly \, dx \, dy$$

$$+ i \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\left(\frac{x}{\sigma}\right)^2} e^{-\left(\frac{y}{\sigma}\right)^2} \sin kx \cos ly \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\left(\frac{x}{\sigma}\right)^2} e^{-\left(\frac{y}{\sigma}\right)^2} \cos kx \sin ly \, dx \, dy \right\}$$

The last three integrals can be evaluated by inspection.  $e^{-\left(\frac{x}{\sigma}\right)^2}$  is an even function ( $f(x) = f(-x)$ ) while the sine is an odd function ( $f(x) = -f(-x)$ ). Notice that the product of an even function and an odd function is an odd function. For any odd function  $g(x)$   $\int_{-\infty}^{\infty} g(x) \, dx = 0$  because the negative contribution from  $-\infty$  to 0 exactly cancels the positive contribution from 0 to  $\infty$ . Since the integrals over  $x$  and  $y$  can be pulled out from one another, the last three integrations always have an integral  $\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sigma}\right)^2} \sin kx \, dx$  or  $\int_{-\infty}^{\infty} e^{-\left(\frac{y}{\sigma}\right)^2} \sin ly \, dy$  both of which must be zero by the preceding arguments. Therefore, using a table of integrals,

$$\hat{f}(0,0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} e^{-\left(\frac{x}{\sigma}\right)^2} e^{-\left(\frac{y}{\sigma}\right)^2} \cos kx \cos ly \, dx \, dy$$

$$\hat{f}(0,0) = \frac{1}{\pi\sigma^2} (\sigma\sqrt{\pi}) (\sigma\sqrt{\pi}) \exp\left(\frac{-k^2\sigma^2}{4}\right) \exp\left(\frac{-l^2\sigma^2}{4}\right)$$

Since  $\hat{f}(0,0)$  is 1 and  $R$  (the response function) is  $R = \frac{\hat{f}}{f}$  it must be that

$$R(k,l) = \exp\left[-\left(\frac{\sigma^2 k^2 + \sigma^2 l^2}{4}\right)\right] \quad (4)$$

As the wave numbers  $k, l \rightarrow 0$ ,  $R(k,l) \rightarrow 1$ . This filter passes the long waves without modification.  $\sigma$  can be chosen on the basis of a desired response for specific wave numbers  $k, l$ . Note that for a given  $k$  and  $l$ , as  $\sigma$

gets larger (analogous to including more points in a discrete application) the response function R gets smaller.

To apply the Gaussian weighting function in the discrete case, the continuous function might be averaged over an area or cell surrounding the grid point. The number of grid points to be included must be based on the value of  $\sigma$ . Let  $k$ ,  $\omega$  and  $\sigma$  be in grid units. Say that the 4ax wave is to have  $1/e$  of its original amplitude. This leads to

$$e^{-1} = \exp\left[-\left(\frac{\sigma^2 \left(\frac{2\pi}{4}\right)^2 + \sigma^2 \left(\frac{2\pi}{4}\right)^2}{4}\right)\right]$$

$$e^{-1} = \exp\left(-\sigma^2 \left(\frac{\pi^2}{8}\right)\right)$$

$$\sigma^2 \frac{\pi^2}{8} = 1$$

$$\sigma^2 = \frac{8}{\pi^2}$$

$$\sigma = \frac{2\sqrt{2}}{\pi}$$

$$\frac{2 \frac{4 \cdot \pi^2}{16 \cdot 8}}{64} = \frac{8 \pi^2}{64}$$

(5)

Now the response function is determined for all wave numbers. The weighting function is

$$w(i, j | i_0, j_0) = \left(\frac{1}{4}\right) e^{-\frac{\pi^2}{8}(i-i_0)^2} e^{-\frac{\pi^2}{8}(j-j_0)^2} \quad (6)$$

should be  $\frac{\pi^2}{8}$

Since the integral  $\int_{i-1/2}^{i+1/2} \left(\frac{1}{4}\right) e^{-\frac{\pi^2}{8}(i-i_0)^2} di$  cannot be evaluated analytically, the

average value of  $w(i, j | i_0, j_0)$  is taken to be the value at the grid point  $i, j$ .

For this discrete case the value of the function at  $(i_0, j_0)$  is taken not to be  $\left(\frac{1}{4}\right) \frac{\pi^2}{8}$  but  $1 - \sum_{\substack{i=i_0 \\ j=j_0}}^{N-1} w_{ij}$  where  $N$  is the number of weights. This is done

to assure that the weights sum to 1. How large should  $N$  be? The larger  $N$  is, the closer the actual response function comes to the analytical function given by (6).

The difference between  $1 - \sum_{\substack{i=i_0 \\ j=j_0}}^{N-1} w_{ij}$  and  $w(i_0, j_0 | i_0, j_0)$  (in this case  $\left(\frac{1}{4}\right) \frac{\pi^2}{8}$ ) is a measure

of how close the response function will be to the analytical one.

The array of points affecting any given grid point should be patterned in a circular sort of way as nearly as possible. In some applications, it is convenient that this array be a square block of grid points. The array dimensions must be an odd number (e.g.,  $3 \times 3$ ,  $5 \times 5$ ,  $7 \times 7$ , etc.) to allow symmetry about  $(i_0, j_0)$ . Once an array is chosen the weights can be computed using (6) in this case. Consider a  $5 \times 5$  array. The weights are



$$w(i_0, j_0) = ? \quad \text{should be } .25$$

$$w(i_0+1, j_0) = .1140 \quad (+)$$

$$w(i_0+2, j_0) = .0108 \quad (+)$$

$$w(i_0+1, j_0+1) = .0520 \quad (+)$$

$$w(i_0+2, j_0+1) = .0049 \quad (+)$$

$$w(i_0+3, j_0+2) = .0005 \quad (+)$$

$$w(i_0+1, j_0+2) = .0049 \quad (+)$$

$$1 - \sum w = .2517 \quad \text{which is close to what it "should" be.}$$

This filter has the following response functions (approximately)

$k$	$L$	$R$
$\frac{2\pi}{24x}$	$\frac{2\pi}{24x}$	.1353
$\frac{2\pi}{48x}$	$\frac{2\pi}{48x}$	.3679 = $e^{-1}$
$\frac{2\pi}{84x}$	$\frac{2\pi}{84x}$	.6065
$\frac{2\pi}{168x}$	$\frac{2\pi}{168x}$	.7788
$\infty$	$\infty$	1.0

$$R(k, l) = \exp\left[-\left(\frac{\sigma^2 k^2 + \sigma^2 l^2}{4}\right)\right]$$

$$R = e^{-1} = \exp\left[-\frac{\sigma^2}{4}(k^2 + l^2)\right]$$

$$-1 = -\frac{\sigma^2}{4}(k^2 + l^2)$$

$$\sigma^2 = \frac{4}{k^2 + l^2}$$

$$\sigma = \frac{2}{\sqrt{k^2 + l^2}}$$

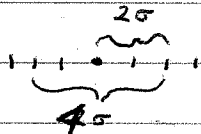
let  $k = l = \frac{2\pi}{n\Delta x}$  where  $\Delta x = \text{grid spacing}$

$$\sigma = \frac{2}{\sqrt{2\left(\frac{4\pi^2}{n^2\Delta x^2}\right)}} = \frac{2}{\frac{2\pi}{n\Delta x} \sqrt{2}}$$

$$\sigma = \frac{n\Delta x}{\pi\sqrt{2}}$$

$$\sigma = \frac{n}{\pi\sqrt{2}} (\Delta x)$$

Use  $2\sigma$  as filter window.



Number grid points to go out is  $\frac{2n}{\pi\sqrt{2}}$

The filter parameter becomes  $n$ , the wave length ( $n\Delta x$ ) at which the response is  $e^{-1} = .3679$

~~The filter parameter becomes  $n$ , the wave length ( $n\Delta x$ ) at which the response is  $e^{-1} = .3679$~~

In  $n \times n$  units  $\sigma = \frac{n}{\pi\sqrt{2}}$

In terms of grid indices with  $\sigma = \frac{n}{\pi\sqrt{2}}$ , the weights are

$$w(i, j | i_0, j_0) = \frac{1}{\sigma^2 \pi} \exp \left[ - \left( \frac{i-i_0}{\sigma} \right)^2 - \left( \frac{j-j_0}{\sigma} \right)^2 \right]$$

$$w(i, j | i_0, j_0) = \frac{2\pi}{n^2} \exp \left[ - \left( \frac{2\pi^2}{n^2} \right) \left( (i-i_0)^2 + (j-j_0)^2 \right) \right]$$

$$\max (i-i_0) = \frac{2n}{\pi\sqrt{2}} \quad \max (j-j_0) = \frac{2n}{\pi\sqrt{2}}$$

$$\text{Compute } w(i_0, j_0) = 1 - \sum_{\substack{i=i_{\min} \\ j=j_{\min}}}^{i=i_{\max} \\ j=j_{\max}} w(i, j | i \neq i_0, j \neq j_0)$$

Calculus

DEFINITE INTEGRALS (Continued)

$$674. \int_0^{\infty} x e^{-ax} [\sin (bx)] dx = \frac{2ab}{(a^2 + b^2)^2}, \quad (a > 0)$$

$$675. \int_0^{\infty} x e^{-ax} [\cos (bx)] dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}, \quad (a > 0)$$

$$676. \int_0^{\infty} x^n e^{-ax} [\sin (bx)] dx = \frac{n! [(a + ib)^{n+1} - (a - ib)^{n+1}]}{2i(a^2 + b^2)^{n+1}}, \quad (i^2 = -1, a > 0)$$

$$677. \int_0^{\infty} x^n e^{-ax} [\cos (bx)] dx = \frac{n! [(a - ib)^{n+1} + (a + ib)^{n+1}]}{2(a^2 + b^2)^{n+1}}, \quad (i^2 = -1, a > 0)$$

$$678. \int_0^{\infty} \frac{e^{-ax} \sin x}{x} dx = \cot^{-1} a, \quad (a > 0)$$

$$\rightarrow 679. \int_0^{\infty} e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{4a^2}\right), \quad (ab \neq 0)$$

$$680. \int_0^{\infty} e^{-t \cos \phi} t^{b-1} \sin (t \sin \phi) dt = [\Gamma(b)] \sin (b\phi), \quad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$$

$$681. \int_0^{\infty} e^{-t \cos \phi} t^{b-1} [\cos (t \sin \phi)] dt = [\Gamma(b)] \cos (b\phi), \quad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$$

$$682. \int_0^{\infty} t^{b-1} \cos t dt = [\Gamma(b)] \cos \left(\frac{b\pi}{2}\right), \quad (0 < b < 1)$$

$$683. \int_0^{\infty} t^{b-1} (\sin t) dt = [\Gamma(b)] \sin \left(\frac{b\pi}{2}\right), \quad (0 < b < 1)$$

$$684. \int_0^1 (\log x)^n dx = (-1)^n \cdot n!$$

$$685. \int_0^1 \left(\log \frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}$$

$$686. \int_0^1 \left(\log \frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}$$

$$687. \int_0^1 \left(\log \frac{1}{x}\right)^n dx = n!$$

$$688. \int_0^1 x \log (1 - x) dx = -\frac{1}{4}$$

$$689. \int_0^1 x \log (1 + x) dx = \frac{1}{4}$$

$$690. \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \quad m > -1, n = 0, 1, 2, \dots$$

If  $n \neq 0, 1, 2, \dots$  replace  $n!$  by  $\Gamma(n+1)$ .